Taylor series (lecture notes)

Arkady M. Alt

For any function f which has continuous derivative of order n on the segment $[c,d]$ and derivative of order $n+1$ on (c,d) and for any $a\in(c,d)$ we will definepolynomial $T_n(f;a)(x) := f(a)+\sum\limits_{k=1}^{n}\dfrac{f^{(k)}(a)}{k!}x^k$ which we call Taylor's Polynomial for function f with node a ($\deg T_n(f;a)(x)\le n$ if T_n isn't zero polynomial).

So, we have the correspondence $(f,n,a)\mapsto T_n(f;a)(x)$.

If f infinitely times differentiable then we get the infinite sequence of Taylor's polynomials:

$T_0(f;a)(x)=f(a)$, $T_1(f;a)(x)=f(a)+\dfrac{f'(a)}{1!}(x-a)$, $T_2(f;a)(x)=$
$f(a)+\dfrac{f'(a)}{1!}(x-a)+\dfrac{f''(a)}{2!}(x-a)^2,\ldots,$

$T_n(f;a)(x)=f(a)+\dfrac{f'(a)}{1!}(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+\ldots+\dfrac{f^{(n)}(a)}{n!}(x-a)^n,\ldots.$

where $T_n(f;a)(x)$ can be considered as partial sum of the series (infinite formal sum) $T(f;a)(x):=f(a)+\sum\limits_{n=1}^{\infty}\dfrac{f^{(k)}(a)}{k!}(x-a)^k=$
$\sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(a)}{n!}(x-a)^n$ (here $f^{(0)}(a)=f(a)$) which we call Taylor series for function f and point a.

Now the two natural questions:

1. What is condition of convergence of this series;
2. When this infinite sum equal to $f(x)$ (in the case of convergence).

Let $r_n(x):=f(x)-T_n(f;a)(x)$ .If $\lim\limits_{n\to\infty} r_n(x)=0$ then $T(f;a)(x)$ convergence to $f(x)$, that is $f(x)=\lim\limits_{n\to\infty}T_n(f;a)(x)=T(f;a)(x)$.

In the supposition that f is function which has continuous derivative of order n on the segment $[c,d]$ and derivative of order $n+1$ on (c,d) we obtain $r_n^{(m)}(a)=0$ for any $m=1,2,..,n$.

Indeed, since $r_n^{(m)}(x)=f^{(m)}(x)-(T_n(f;a)(x))^{(m)}=$
$f^{(m)}(x)-\dfrac{f^{(m)}(a)}{m!}\cdot m!-(x-a)\sum\limits_{k=m+1}^{n}\left(\dfrac{f^{(k1)}(a)}{k!}\cdot k(k-1)\ldots(k-m+1)(x-a)^{k-1}\right)\implies$
$r_n^{(m)}(a)=0$ for any $m=0,1,,2\ldots,n$.

Definition.

Let $\lim\limits_{x\to a}f(x)=0$ and $\lim\limits_{x\to a}g(x)=0$ then if $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=0$ we say that $f(x)$ in the point a has order of smallness bigger the $g(x)$ and write down that as follows $f(x)=o(g(x))$.

In particular $f(x)=o((x-a)^n)$ if $\lim\limits_{x\to a}\dfrac{f(x)}{(x-a)^n}=0$.

Obvious that:

1. $c\cdot o\left((x-a)^n\right)=o\left(g\left(x\right)\right)$ for any constant c;

2. $\dfrac{o\left((x-a)^n\right)}{(x-a)^k}=o\left((x-a)^{n-k}\right)$,for any $k=1,2,...,n-1$ and if $k=n$ then $\dfrac{o\left((x-a)^n\right)}{(x-a)^n}=o\left(1\right)$ $\left(f\left(x\right)=o\left(1\right)\iff\lim\limits_{x\to a}f\left(x\right)=0\right)$;

3. $(x-a)^k o\left((x-a)^n\right)=o\left((x-a)^{n+k}\right)$ for any $k\in\mathbb{N}$.

4. $o\left((x-a)^n\right)+o\left((x-a)^m\right)=o\left((x-a)^{\min\{n,m\}}\right)$.

Lemma 1 .
Let f differentiable of order $n-1$ in any $x\in(a-\varepsilon,a+\varepsilon)$ for some ε and has derivative of order n in the point a and $f^{(n)}\left(x\right)$ is continuous in a.
Then $f\left(x\right)=o\left((x-a)^n\right)$ iff $f\left(a\right)=f'\left(a\right)=...=f^{(n)}\left(a\right)=0$.
Proof (by Math Induction).

Sufficiency

1. Base of MI.
Let $n=1$. Since tby Mean Value Theorem there is

$c_x\in\overline{(a,x)}$ $\left(\overline{(a,x)}=\begin{cases}(a,x) & \text{if } x>a\\ (x,a) & \text{if } x>a\end{cases}\right)$

such that $\dfrac{f\left(x\right)}{x-a}=\dfrac{f\left(x\right)-f\left(a\right)}{x-a}=f'\left(c_x\right)$ and $\lim\limits_{x\to a}c_x=0$ then

$$\lim_{x\to a}\frac{f\left(x\right)}{x-a}=\lim_{x\to a}f'\left(c_x\right)=f'\left(\lim_{x\to a}c_x\right)=f'\left(a\right)=0$$

(because f' is continuous in a). Thus, $f\left(x\right)=o\left((x-a)\right)$.

2. Step of MI.
Let $f\left(a\right)=f'\left(a\right)=...=f^{(n)}\left(a\right)=f^{(n+1)}\left(a\right)=0$ and $f^{(n+1)}\left(x\right)$ is continuous in a.And let $g\left(x\right):=f'\left(x\right)$. Then

$$f'\left(a\right)=...=f^{(n)}\left(a\right)=f^{(n+1)}\left(a\right)=0\iff g\left(a\right)=g'\left(a\right)=...=g^{(n)}\left(a\right)=0$$

and by supposition of MI we have $g\left(x\right)=o\left((x-a)^n\right)$, i.e.

$\lim\limits_{x\to a}\dfrac{g\left(x\right)}{(x-a)^n}=0$.

Hence, $\lim\limits_{x\to a}\dfrac{f\left(x\right)}{(x-a)^{n+1}}=\lim\limits_{x\to a}\dfrac{f\left(x\right)-f\left(a\right)}{x-a}\cdot\dfrac{1}{(x-a)^n}=\lim\limits_{x\to a}f'\left(c_x\right)\cdot\dfrac{1}{(x-a)^n}=$
$\lim\limits_{x\to a}\left(\dfrac{f'\left(c_x\right)}{(c_x-a)^n}\cdot\dfrac{(c_x-a)^n}{(x-a)^n}\right)=\lim\limits_{x\to a}\left(\dfrac{g\left(c_x\right)}{(c_x-a)^n}\cdot\left(\dfrac{c_x-a}{x-a}\right)^n\right)=0$

because $\lim\limits_{x\to a}c_x=0\implies\lim\limits_{x\to a}\dfrac{g\left(c_x\right)}{(c_x-a)^n}=\lim\limits_{c_x\to a}\dfrac{g\left(c_x\right)}{(c_x-a)^n}=0$

and $\left|\dfrac{c_x-a}{x-a}\right|<1$. So, $\lim\limits_{x\to a}\dfrac{f\left(x\right)}{(x-a)^{n+1}}=0$.

Necessity.
Let $n\in\mathbb{N}$ and $f\left(x\right)=o\left((x-a)^n\right)$. Obviously that $f\left(a\right)=0$ and

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{f(x)}{x-a}\ =\lim_{x\to a}\frac{o((x-a)^n)}{x-a}=\lim_{x\to a}o\left((x-a)^{n-1}\right)=0.$$

Since $r_n(a)=r_n'(a)=...=r_n^{(n)}(a)=0$ then $r_n(x)=o((x-a)^n)$
and we obtain

$$o((x-a)^n)=f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+...+\frac{f^{(n)}(a)}{n!}(x-a)^n+r_n(x)\iff$$

$$\textbf{(1)}\quad f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+...+\frac{f^{(n)}(a)}{n!}(x-a)^n=o((x-a)^n)-r_n(x)=o((x-a)^n)$$

.

Passing in **(1)** to the limit when $x\to a$ we obtain $f(a)=0$.
Then

$$\frac{f'(a)}{1!}+\frac{f''(a)}{2!}(x-a)+...+\frac{f^{(n)}(a)}{n!}(x-a)^{n-1}=\frac{o((x-a)^n)}{x-a}=o\left((x-a)^{n-1}\right)\implies f'(a)=0$$

and so on ... For any $k<n$ assuming $f(a)=f'(a)=...=f^{(k)}(a)=0$
we obtain

$$\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}+...+\frac{f^{(n)}(a)}{n!}(x-a)^n=o((x-a)^n)\iff$$

$$\frac{f^{(k+1)}(a)}{(k+1)!}+...+\frac{f^{(n-k-1)}(a)}{n!}(x-a)^n=o\left((x-a)^{n-k-1}\right)\implies f^{(k+1)}(a)=0.$$

Let f infinitely times differentiable in a. Then $f^{(n)}(x)$ is continuous in a for any $n\in\mathbb{N}$ and now we can apply this Lemma to

$r_n(x)=f(x)-T_n(f;a)(x)$ and obtain $r_n(x)=o((x-a)^n)$

Thus, in that case $f(x)=T_n(f;a)(x)+o((x-a)^n)$.(It is Polynomial Taylor representation of $f(x)$ with error in Peano form or, shortly Peano form of Taylor representation for $f(x)$).

Corollary from Lemma 1 .
Let f differentiable of order $n-1$ in any $x\in(a-\varepsilon,a+\varepsilon)$ for some ε and has derivative of order n in the point a and $f^{(n)}(x)$ is continuous in a.
Then function $g(x)$, such that $g(x)$ is continuous on $(a-\varepsilon,a+\varepsilon)$ and $f(x)=(x-a)^n g(x)$ holds in $(a-\varepsilon,a+\varepsilon)$ exists iff

$f(a)=f'(a)=...=f^{(n)}(a)=0.$

Proof.
Sufficiency

Let $f(a)=f'(a)=...=f^{(n)}(a)=0$ and let $g(x):=\begin{cases}\dfrac{f(x)}{(x-a)^n}, x\neq a\\ 0 \text{ if } x=a\end{cases}$.

Since $\lim\limits_{x\to a}\dfrac{f(x)}{(x-a)^n}=0$ then $g(x)$ defined by such way is continuous in a.

Necessity.

let $f(x)=(x-a)^n g(x)$,where g is continuous in a.Then

$$f(x)=(x-a)^n g(x) \iff \frac{f(x)}{(x-a)^n}=g(x) \implies$$

$$\lim_{x\to a}\frac{f(x)}{(x-a)^n}=0 \iff f(x)=o((x-a)^n) \implies f(a)=f'(a)=...=f^{(n)}(a)=0.\blacksquare$$

Since $\lim\limits_{x\to a}\dfrac{r_n(x)}{(x-a)^n}=0$ then for any $\varepsilon>0$ there is $0<\delta<1$ such that for any $x\in(a-\delta,a+\delta)$ we have $\left|\dfrac{r_n(x)}{(x-a)^n}\right|<\varepsilon \iff |r_n(x)|<\varepsilon\delta^n \implies$ $\lim\limits_{n\to\infty} r_n(x)=0$ and, therefore,

$$\lim_{n\to\infty} T_n(f;a)(x)=f(x) \iff f(x)=T(f;a)(x)$$

for any $x\in(a-\delta,a+\delta)$.

Peano form wery convenient for finding limits, but more information of error $r_n(x)$ give Lagrange form.

Let $f(x)$ has on $[p,q]$ continuous derivative of order n and derivative of order $n+1$ on interval (p,q).

For fixed $x\in(p,q)$ and any $t\in(p,q)$ we will find constant K (not depends from t) such that

$$r_n(x)=f(t)-T_n(f;a)(t)=K(x-a)^{n+1}\text{,that is }K:=\frac{r_n(x)}{(x-a)^{n+1}}.$$

Then for any $t\in(p,q)$ denote $\varphi(t):=f(t)-T_n(f;a)(t)-K(t-a)^{n+1}$, which obviously $n+1$ time differentiable on (p,q) we have $\varphi(a)=\varphi'(a)=...=\varphi^{(n)}(a)=0$ and $\varphi(x)=0$ by definition of K.

Since $\varphi(a)=\varphi(x)=0$ then by Roll Theorem there is $c_1\in\overline{(a,x)}$ such that $\varphi'(c_0)=0$.

Then again by Roll Theorem there is $c_2\in\overline{(a,c_1)}\subset\overline{(a,x)}$ such that $\varphi''(c_2)=0$.

Assume that we already has $c_k\in\overline{(a,x)}$ such that $\varphi^{(k)}(c_k)=0, k<n$.

Then, since $\varphi^{(k)}(c_k)=\varphi^{(k)}(a)=0$ we obtain by Roll Theorem $\varphi^{(k+1)}(c_{k+1})=0$ for some $c_{k+1}\in\overline{(a,c_k)}\subset\overline{(a,x)}$.

Thus we finally obtain $\varphi^{(n)}(c_n)=\varphi^{(n)}(a)=0$ and, therefore, by Roll Theorem there is $c_{n+1}\in\overline{(a,c_n)}\subset\overline{(a,x)}$ such that

$$\varphi^{(n+1)}(c_{n+1})=0 \iff f^{(n+1)}(c_{n+1})-(T_n(f;a))^{(n+1)}(c_{n+1})-K\left((t-a)^{n+1}\right)^{(n+1)}=0 \iff$$

$$f^{(n+1)}(c_{n+1})-K(n+1)!=0 \iff K=\frac{f^{(n+1)}(c_{n+1})}{(n+1)!}$$

.

Since $c_{n+1} \in \overline{(a,x)}$ then denoting $\theta := \dfrac{c_{n+1}-a}{x-a} \in (0,1)$ we obtain
$c_{n+1} = a(1-t) + xt = a + \theta(x-a)$.
Hence, $K = \dfrac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}, \theta \in (0,1)$ and, therefore,

$$r_n(x) = \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1} \iff f(x) = T_n(f;a)(x) + \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}$$

(Polynomial Taylor fepresentation of $f(x)$ with error in Lagrange form).
Denoting $h := x-a$ we we obtain another form of Taylor representation for f,namely,

$$f(a+h) = \sum_{k=0}^{n} f^{(k)}(a) h^k + \frac{f^{(n+1)}(a+\theta h)}{(n+1)!} h^{n+1}.$$

If $M := \sup\limits_{x\in(p,q)} \left|f^{(n+1)}(x)\right|$ then for any $x \in (p,q)$ we have

$$|r_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Deriving Taylor formula with error $r_n(x)$ integral form (using integration by parts):

By Newton-Leybnitz formula $f(x) - f(a) = \int_a^x f'(t)\,dt = \begin{bmatrix} u' = -1; u = x-t \\ v = -f'(t); v' = -f^{(2)}(t) \end{bmatrix} =$
$(-f'(t)(x-t))_a^x + \int_a^x (x-t) f^{(2)}(t)\,dt = f'(a)(x-a) + \int_a^x (x-t) f^{(2)}(t)\,dt =$
$\begin{bmatrix} u' = -(x-t); u = \dfrac{(x-t)^2}{2} \\ v = -f^{(2)}(t); v' = -f^{(3)}(t) \end{bmatrix} = f'(a)(x-a) + \left(-f^{(2)}(t)\dfrac{(x-t)^2}{2!}\right)_a^x +$
$\dfrac{1}{2!}\int_a^x (x-t)^2 f^{(3)}(t)\,dt = f'(a)(x-a) + f^{(2)}(a)\dfrac{(x-a)^2}{2!} + \dfrac{1}{2!}\int_a^x (x-t)^2 f^{(3)}(t)\,dt.$
Assume that we already have

$f(x) - f(a) = f'(a)(x-a) + f^{(2)}(a)\dfrac{(x-a)^2}{2!} + \dots + f^{(k)}(a)\dfrac{(x-a)^k}{k!} +$
$\dfrac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$

then using ntegration by parts again we obtain

$\int_a^x (x-t)^k f^{(k+1)}(t)\,dt = \begin{bmatrix} u' = -(x-t)^k; u = \dfrac{(x-t)^{k+1}}{k+1} \\ v = -f^{(k+1)}(t); v' = -f^{(k+2)}(t) \end{bmatrix} =$
$\left(-f^{(k+1)}(t)\dfrac{(x-t)^{k+1}}{k+1}\right)_a^x + \dfrac{1}{k+1}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt = f^{(k+1)}(a)\dfrac{(x-a)^{k+1}}{k+1} +$
$\dfrac{1}{k+1}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt.$

Hence,

$f(x) - f(a) = f'(a)(x-a) + f^{(2)}(a)\dfrac{(x-a)^2}{2!} + \dots + f^{(k)}(a)\dfrac{(x-a)^k}{k!} +$

$f^{(k+1)}(a)\frac{(x-a)^{k+1}}{(k+1)!}+\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)\,dt.$

For $k=n$ we obtain

$f(x) = f(a)+f'(a)(x-a)+f^{(2)}(a)\frac{(x-a)^2}{2!}+...+f^{(n)}(a)\frac{(x-a)^n}{n!}+\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt =$

$T_n(f;a)(x)+\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$

So, $r_n(x)=\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$

Using integral Mean Value Theorem we obtain

$\int_a^x (x-t)^n f^{(n+1)}(t)\,dt = f^{(n+1)}(c)\int_a^x (x-t)^n\,dt = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{n+1},$

for some $c\in\overline{(a,x)}$.

Therefore, $f(x)=T_n(f;a)(x)+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$

Lemma 2.

If $a_0+a_1(x-a)+...+a_n(x-a)^n = o((x-a)^n)$ then $a_0=a_1=...a_n=0.$

Proof.

$$\lim_{x\to a}(a_0+a_1(x-a)+...+a_n(x-a)^n)=\lim_{x\to a} o((x-a)^n)=0 \implies a_0=0.$$

Let $k<n$. Assuming $a_0=a_1=...a_k=0$ we obtain

$a_{k+1}(x-a)^{k+1}+...+a_n(x-a)^n = o((x-a)^n) \iff$

$$a_{k+1}+a_{k+2}(x-a)+...+a_n(x-a)^{n-k+1} = o\left((x-a)^{n-k+1}\right) \implies a_{k+1}=0.$$

Hence, by MI we proved $a_0=a_1=...a_n=0.$

Corollary1.

If

$$a_0+a_1(x-a)+...+a_n(x-a)^n+o((x-a)^n)=b_0+b_1(x-a)+...+b_n(x-a)^n+o((x-a)^n)$$

then $a_k=b_k, k=1,2,...,n.$

Proof.

$$a_0+a_1(x-a)+...+a_n(x-a)^n+o((x-a)^n)=b_0+b_1(x-a)+...+b_n(x-a)^n+o((x-a)^n) \iff$$

$$(a_0-b_0)+(a_1-b_1)(x-a)+...+(a_n-b_n)(x-a)^n=o((x-a)^n) \implies a_k=b_k, k=1,2,...,n.$$

Corollary 2.

If $f(x) = a_0 + a_1 (x-a) + ... + a_n (x-a)^n + o((x-a)^n)$ then
$a_k = \frac{f^{(k)}}{k!}, k = 1, 2, ..., n.$

Proof.

Follow fom **Corollary1** and Taylor Representation for $f(x)$ in Peano form.

Applications.

I. Taylor representation for some elementary functions.

a) Let $f(x) = e^x$. Since $f(0) = 1$ and $f^{(n)}(x) = e^x \implies f^{(n)}(0) = 1, n \in \mathbb{N}$ then

$e^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!} + r_n(x),$ where $r_n(x) = \frac{e^{\theta x} x^{n+1}}{(n+1)!}$ and $\theta \in (0,1)$.

For any fixed real x we have $\lim_{x \to 0} \frac{r_n(x)}{x^n} = 0$ and $\lim_{n \to \infty} r_n(x) = 0$,that is $T(e^x; 0)(x)$
convergent for any real x.

Thus, $T(e^x; 0)(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \; e^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!} + o(x^n)$ and since
$\left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| = \frac{e^{\theta x} |x|^{n+1}}{(n+1)!} < \frac{e|x|^{n+1}}{(n+1)!}$then $|r_n(x)| < \frac{e|x|^{n+1}}{(n+1)!}$.

(If $x < 0$ then $\left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| = \frac{e^{\theta x} |x|^{n+1}}{(n+1)!} < \frac{e^0 |x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$ and this inequality
convenient for estimation of error of Taylor approximation for e^x).

b) Let $f(x) = \sin x$. Then $f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right), \; f''(x) = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + 2 \cdot \frac{\pi}{2}\right)$.Assuming $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ we obtain

$$f^{(n+1)}(x) = \left(\sin\left(x + \frac{n\pi}{2}\right)\right)' = \cos\left(x + \frac{n\pi}{2}\right) = \sin\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + \frac{(n+1)\pi}{2}\right).$$

Thus, by MI we proved that $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right), n \in \mathbb{N}$.

Hence, $f^{(n)}(0) = \sin \frac{n\pi}{2} = \begin{cases} 0 \text{ if } n \text{ even} \\ 1 \text{ if } rem_4 n = 1 \\ -1 \text{ if } rem_4 n = 3 \end{cases}$ and, therefore,

$T_{2n-1}(f; 0)(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, T(f; 0)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!},$

$r_{2n-1}(x) = r_{2n}(x) = \frac{\sin\left(\theta + \frac{(2n+1)\pi}{2}\right) x^{2n+1}}{(2n+1)!} = \frac{\cos(\theta x + n\pi) x^{2n+1}}{(2n+1)!} = o(x^{2n})$.

Since $|\cos(\theta + n\pi)| \leq 1$ then $|r_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$. So, $T(f;0)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$ convergence to $\sin x$ for any real x.

c) Let $f(x) = \cos x$. Then $f'(x) = -\sin = \cos x\left(x + \frac{\pi}{2}\right)$, $f''(x) = -\sin\left(x + \frac{\pi}{2}\right) =$

$\cos\left(x + \frac{\pi}{2} + \frac{\pi}{2}\right) = \cos\left(x + 2 \cdot \frac{\pi}{2}\right)$ .Assuming $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$ we obtain

$f^{(n+1)}(x) = \left(\cos\left(x + \frac{n\pi}{2}\right)\right)' = -\sin\left(x + \frac{n\pi}{2}\right) = \cos\left(x + \frac{n\pi}{2} + \frac{\pi}{2}\right) = \cos\left(x + \frac{(n+1)\pi}{2}\right)$.

Thus, by MI we proved that $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right), n \in \mathbb{N}$.

Hence, $f^{(n)}(0) = \cos\frac{n\pi}{2} = \begin{cases} 0 \text{ if } n \text{ odd} \\ 1 \text{ if } rem_4 n = 0 \\ -1 \text{ if } rem_4 n = 2 \end{cases}$ and, therefore,

$T_{2n}(f;0)(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^{n-1} \frac{x^{2n}}{(2n)!}$, $T(f;0)(x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n)!}$,

$r_{2n}(x) = r_{2n+1}(x) = \frac{\cos\left(\theta x + \frac{(2n+1)\pi}{2}\right) x^{2n+2}}{(2n+2)!} = \frac{\cos(\theta x + n\pi) x^{2n+2}}{(2n+1)!} = o\left(x^{2n+1}\right)$.

Since $|\cos(\theta + n\pi)| \leq 1$ then $|r_{2n}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$. So, $T(f;0)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n)!}$ convergence to $\sin x$ for any real x.

d) Let $f(x) = \ln(1+x)$ .Then $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f^{(3)}(x) = \frac{2}{(1+x)^3}$,

$f^{(4)}(x) = -\frac{2 \cdot 3}{(1+x)^4}$, $f^{(5)}(x) = \frac{2 \cdot 3 \cdot 4}{(1+x)^5}$, ..., $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$ (Prove that by MI)

Hence, $f(0) = 0$, $f^{(n)}(0) = \frac{(-1)^{n-1}}{n}$ and, therefore,

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{(-1)^{n-1} x^n}{n} + ..$ or $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{(-1)^{n-1} x^n}{n} + r_n(x)$

where $r_n(x) = o(x_n) = (-1)^{n-1} \frac{n!}{(1+\theta x)^{n+1}}$, $\theta \in (0,1)$.

Remark. Taylor series for $\ln(1-x)$ without derivatives.

Let $S_n(x) := 1 + x + ... + x^{n-1} = \dfrac{1 - x^n}{1 - x}, x \neq 1$. Since for any $x \in [0, 1)$ we have

$\lim\limits_{n\to\infty} \left(\dfrac{1}{1-x} - S_n(x)\right) = \lim\limits_{n\to\infty} \dfrac{x^n}{1-x} = 0$ then $\sum\limits_{n=1}^{\infty} x^{n-1} = \dfrac{1}{1-x}$.

We will prove that $\ln(1-x) = -\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}, x \in [0,1)$ that is

$-\ln(1-x) = \lim\limits_{n\to\infty} \int_0^x S_n(t)\,dt$.

We have $\int_0^x \left(\dfrac{1}{1-t} - S_n(t)\right) dt = \int_0^x \dfrac{t^n}{1-t} dt \iff -\ln(1-t) - \int_0^x S_n(t)\,dt = \int_0^x \dfrac{t^n}{1-t} dt \iff$

$-\ln(1-t) - \sum\limits_{k=1}^{n} \dfrac{x^k}{k} = \int_0^x \dfrac{t^n}{1-t} dt.$

Since $\int_0^x t^n dt < \int_0^x \dfrac{t^n}{1-t} dt < \int_0^x \dfrac{t^n}{1-x} dt \iff \dfrac{x^{n+1}}{n+1} < \int_0^x \dfrac{t^n}{1-t} dt < \dfrac{x^{n+1}}{(1-x)(n+1)} \iff$

$\dfrac{x^{n+1}}{n+1} < -\ln(1-t) - \sum\limits_{k=1}^{n} \dfrac{x^k}{k} < \dfrac{x^{n+1}}{(1-x)(n+1)}$ and $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{n+1} = \lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(1-x)(n+1)} = 0$

then by Squeeze Principle $\lim\limits_{n\to\infty} \left(-\ln(1-t) - \sum\limits_{k=1}^{n} \dfrac{x^k}{k}\right) = 0 \iff$

$\ln(1-t) = \lim\limits_{n\to\infty} \left(-\sum\limits_{k=1}^{n} \dfrac{x^k}{k}\right) = -\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}$

e) Let $f(x) = (1+x)^{\alpha}$, where $\alpha \in \mathbb{R} \diagdown \mathbb{N} \cup \{0\}$. Since $f^{(n)}(x) = \alpha(\alpha-1)...(\alpha-n+1)(1+x)^{\alpha-n}$ then $\dfrac{f^{(n)}(0)}{n!} = \dfrac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}$ and denoting $\dbinom{\alpha}{n} := \dfrac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}$ (like binomial coefficients)

we obtain $(1+x)^{\alpha} = \sum\limits_{n=0}^{\infty} \dbinom{\alpha}{n} x^n$ or,

$(1+x)^{\alpha} = \sum\limits_{k=0}^{n} \dbinom{\alpha}{k} x^k + \dbinom{\alpha}{n+1} (1+\theta x)^{\alpha-n-1} x^{n+1} = \sum\limits_{k=0}^{n} \dbinom{\alpha}{k} x^k + o(x^n)$ (binomial series).

Remark.

Some times calculation $f^{(n)}(0)$ became hard problem or even imposible because can be performed

throug calculation $f^{(n)}(x)$. For example if $f(x) = \arctan x$ then

$f'(x) = \dfrac{1}{1+x^2}, f''(x) = \left(\dfrac{1}{1+x^2}\right)' = \dfrac{-2x}{(x^2+1)^2}, f^{(3)}(x) = \left(\dfrac{-2x}{(x^2+1)^2}\right)' = \dfrac{23x^2-1}{(x^2+1)^3}$, and so on ..

We can see that complexity grow up.

Problem.

Find Taylor series for $f(x) = \arctan x, \arcsin x, \ln \dfrac{1+x}{1-x}, \ln \dfrac{1+x+x^2}{1-x+x^2}$ (use the following properties

of Taylor operator defined as follows: $(f, a) \longmapsto T_n(f)(x) = \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(x-a)^k$).

Properties of $T_n(f)$

1. $T_n(f+g) = T_n(f) + T_n(g)$;
2. $T_n(cf) = cT_n(f)$;
3. $D_x(T_n(f)) = T_{n-1}(f')$;
4. $\int_a^x T_n(f)(t)\,dt = T_{n+1}(F)(x)$, where $F(x) = \int_a^x f(t)\,dt$.

Proof.

We have:

1 $T_n(f+g)(x) = \sum\limits_{k=0}^{n} \dfrac{(f+g)^{(k)}(a)}{k!}(x-a)^k = \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a) + g^{(k)}(a)}{k!}(x-a)^k =$

$\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(x-a)^k + \sum\limits_{k=0}^{n} \dfrac{g^{(k)}(a)}{k!}(x-a)^k = T_n(f)(x) + T_n(g)(x) = (T_n(f) + T_n(g))(x)$,

2.$T_n(cf)(x) = \sum\limits_{k=0}^{n} \dfrac{(cf)^{(k)}(a)}{k!}(x-a)^k = \sum\limits_{k=0}^{n} \dfrac{cf^{(k)}(a)}{k!}(x-a)^k = c\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(x-a)^k =$ $cT_n(f)(x)$

3. $D_x(T_n(f)) = \left(\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(x-a)^k\right)' = \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}\left((x-a)^k\right)' =$

$\sum\limits_{k=1}^{n} \dfrac{f^{(k)}(a)}{(k-1)!}(x-a)^{k-1} =$

$\sum\limits_{k=0}^{n-1} \dfrac{f^{(k+1)}(a)}{(k-1)!}(x-a)^k = T_{n-1}(f')(x)$.

4. Let $F(x) = \int_a^x f(t)\,dt$ then $F(a) = 0$ and $\int_a^x T_n(f)(t)\,dt = \int_a^x \left(\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}(t-a)^k\right)dt =$

$\sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{k!}\int_a^x (t-a)^k\,dt = \sum\limits_{k=0}^{n} \dfrac{f^{(k)}(a)}{(k+1)!}(x-a)^{k+1} = \sum\limits_{k=1}^{n+1} \dfrac{f^{(k-1)}(a)}{k!}(x-a)^k =$

$\sum\limits_{k=1}^{n+1} \dfrac{F^{(k)}(a)}{k!}(x-a)^k$

Note that $o((x-a)^n) + o((x-a)^n) = o((x-a)^n)$, $o((x-a)^n) = o((x-a)^n)$,

$(o((x-a)^n))' = o\left((x-a)^{n-1}\right)$.

Lemma 3.

Let $\varphi(x) = o((x-a)^n)$ and $\varphi(a) = \varphi'(a) = ... = \varphi^{(n)}(a) = 0$.

Then $\int_a^x \varphi(t)\,dt = o\left((x-a)^{n+1}\right)$.

Proof.

Since $\varphi(x) := o((x-a)^n)$ and $\varphi(a) = \varphi'(a) = ... = \varphi^{(n)}(a) = 0$ then

$\varphi(x) = g(x)(x-a)^n$ and, therefore, $\int_a^x o((t-a)^n)\,dt = \int_a^x g(t)(t-a)^n\,dt =$

$g(c_x)\int_a^x (t-a)^n\,dt = \dfrac{g(c_x)}{n+1}(x-a)^{n+1} =$

$$\frac{1}{n+1}\cdot\frac{g(c_x)(c_x-a)^n}{(c_x-a)^n}\cdot(x-a)^{n+1}=\frac{1}{n+1}\cdot\frac{\varphi(c_x)}{(c_x-a)^n}\cdot(x-a)^{n+1}.$$

Hence, $\lim\limits_{x\to a}\dfrac{\int_a^x o\left((t-a)^n\right)dt}{(x-a)^{n+1}}=\dfrac{1}{n+1}\lim\limits_{x\to a}\dfrac{\varphi(c_x)}{(c_x-a)^n}=0\implies\int_a^x o\left((t-a)^n\right)dt=o\left((t-a)^{n+1}\right).$

Problems.

1. Find limits.

a) $\lim\limits_{x\to0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$; **b)** $\lim\limits_{x\to0}\dfrac{a^x+a^{-x}-2}{x^2}$; **c)** $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$;

d) $\lim\limits_{x\to0}\dfrac{e^x\sin x-x(1+x)}{x^3}$; **d)** $\lim\limits_{x\to0}\dfrac{1}{x}\left(\dfrac{1}{x}-\cot x\right)$; **e)** $\lim\limits_{x\to0}\left(x-x^2\ln\left(1+\dfrac{1}{x}\right)\right)$.

2. For which a,b holds $x-(a+b\cos x)\sin x=o\left(x^5\right)$.

Estimate errors of the following approximations:

3. **a)** $\sin x\approx x-\dfrac{x^3}{6},|x|\le\dfrac{1}{2}$; **b)** $\tan x\approx x+\dfrac{x^3}{6},\ \ |x|\le0.1$;

c) $\sqrt{1+x}\approx1+\dfrac{x}{2}-\dfrac{x^2}{8}$.

4. For which x holds $\left|\cos x-\left(1-\dfrac{x^2}{2}\right)\right|<0.0001$.

Additional problems with solutions.

1. Sum of one power series.

Find the sum $\sum\limits_{n=1}^{\infty}\dfrac{(2n-1)!!}{(2n)!!}x^n$.

Solution 1.

Let $S(x):=\sum\limits_{n=1}^{\infty}\dfrac{(2n-1)!!}{(2n)!!}x^n$. Since Taylor series for $\dfrac{1}{\sqrt{1-x}}=(1-x)^{-1/2}=1+\sum\limits_{n=1}^{\infty}\dbinom{-1/2}{n}(-x)^n$ and

$$\binom{-1/2}{n}=\frac{(-1/2)(-1/2-1)\dots(-1/2-n+1)}{n!}=\frac{(-1)^n(2n-1)!!}{2^n n!}=\frac{(-1)^n(2n-1)!!}{(2n)!!}$$

then $\dfrac{1}{\sqrt{1-x}}=1+\sum\limits_{n=1}^{\infty}\dfrac{(2n-1)!!}{(2n)!!}x^n$ and, therefore, $S(x)=\dfrac{1}{\sqrt{1-x}}-1$.

Solution 2. (Direct, without using Taylor expansion for $\dfrac{1}{\sqrt{1-x}}$).

Let $T(x)=\sum\limits_{n=1}^{\infty}\dfrac{(2n-1)!!}{(2n-2)!!}x^{n-1}$. Since $\dfrac{(2n+1)!!}{(2n)!!}=\dfrac{(2n-1)!!\cdot2n}{(2n)!!}+\dfrac{(2n-1)!!}{(2n)!!}=\dfrac{(2n-1)!!}{(2n-2)!!}+\dfrac{(2n-1)!!}{(2n)!!}$ then $\dfrac{(2n-1)!!}{(2n)!!}=\dfrac{(2n+1)!!}{(2n)!!}-\dfrac{(2n-1)!!}{(2n-2)!!}$ and

$$S(x)=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^n=\sum_{n=1}^{\infty}\frac{(2n+1)!!}{(2n)!!}x^n-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-2)!!}x^n=$$

$$\sum_{n=1}^{\infty}\frac{(2n+1)!!}{(2n)!!}x^n-x\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-2)!!}x^{n-1}=\sum_{n=2}^{\infty}\frac{(2n-1)!!}{(2n-2)!!}x^{n-1}-x\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-2)!!}x^{n-1}=$$

$T(x)-1-xT(x)=T(x)(1-x)-1$.

Noting that $S'(x) = \frac{1}{2}\sum\limits_{n=1}^{\infty}\frac{(2n-1)!!}{(2n-2)!!}x^{n-1} = \frac{1}{2}T(x)$ we obtain
$T(x) = 2S'(x)$ and, therefore,
$S(x) = 2S'(x)(1-x) - 1 \iff S(x)+1 = 2(S(x)+1)'(1-x) \iff$
$\frac{(S(x)+1)'}{S(x)+1} = \frac{1}{2}\cdot\frac{1}{1-x} \iff \ln(S(x)+1) = \frac{1}{2}\ln\left(\frac{1}{1-x}\right) + c.$
Since $\ln(S(0)+1) = \ln(0+1) = 0$ and $\frac{1}{2}\ln\left(\frac{1}{1-0}\right) = 0$ then $c = 0$
and, therefore, $S(x)+1 = \frac{1}{\sqrt{1-x}} \iff S(x) = \frac{1}{\sqrt{1-x}} - 1.$

2. One limit related to Taylor Formula.
Let $f \in C^{n+1}((-1,1))$, $f^{(n+1)}(0) \neq 0, n \geq 1$ and for any $x \in (-1,1)$
the value $\theta_x = \theta_{x,n}$ is determined as number $\theta \in (0,1)$ such that
$f(x) = \sum\limits_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n)}(\theta_x \cdot x)}{n!}x^n$.Find $\lim\limits_{x\to 0}\theta_x$.
Solution.
Since $|\theta_x x| < |x|$ then $\lim\limits_{x\to 0}\frac{f^{(n)}(\theta_x\cdot x) - f^{(n)}(0)}{\theta_x\cdot x} = f^{(n+1)}(0)$.
From the other hand we have $f(x) = \sum\limits_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta_1\cdot x)}{(n+1)!}x^{n+1}$,
where $\theta' = \theta_{x,n+1} \in (0,1)$
Hence,

$$\frac{f^{(n)}(\theta_x\cdot x)}{n!}x^n = \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta_1\cdot x)}{(n+1)!}x^{n+1} \iff f^{(n)}(\theta_x\cdot x) = f^{(n)}(0) + \frac{f^{(n+1)}(\theta_1\cdot x)}{n+1}x \iff$$

$\frac{f^{(n)}(\theta_x\cdot x) - f^{(n)}(0)}{\theta_x\cdot x}\cdot\theta_x = \frac{f^{(n+1)}(\theta_1\cdot x)}{(n+1)}.$
Since $f \in C^{n+1}((-1,1))$ then $\lim\limits_{x\to 0} f^{(n+1)}(\theta_1\cdot x) = f^{(n+1)}(0)$
and, therefore,
$\lim\limits_{x\to 0}\frac{f^{(n)}(\theta_x\cdot x) - f^{(n)}(0)}{\theta_x\cdot x}\cdot\theta_x = \frac{1}{(n+1)}\lim\limits_{x\to 0} f^{(n+1)}(\theta_1\cdot x) \iff$
$\lim\limits_{x\to 0} f^{(n+1)}(0)\lim\limits_{x\to 0}\theta_x = \frac{f^{(n+1)}(0)}{(n+1)\theta} \iff \lim\limits_{x\to 0}\theta_x = \frac{1}{n+1}.$